A sufficient condition for optimality in mutual exclusion scheduling with interval graphs and related classes

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1 Introduction

The following problem arises in scheduling theory: n unit-time jobs must be completed on k processors in a minimum time with the constraint that some jobs cannot be executed at the same time because they share the same resources. Many variants of this problem have been considered in combinatorial optimization and operations research literature (see Blazewicz et al. (1996) for a survey). Such a scheduling problem can be alternatively formulated in graph-theoretic terms: by creating an undirected graph $G = (V, E)$ with a vertex for each of the n jobs and an edge between each pair of conflicting jobs, a schedule of minimum length corresponds to a partition of V into a minimum number of independent sets of size at most k. Therefore, Baker and Coffman (1996) called the following restricted coloring problem MUTUAL EXCLUSION SCHEDULING, shortly MES: given an undirected graph G and an integer k, find a minimum coloring of G such that each color appears at most k times.

Since the classical coloring problem is $\mathcal{NP}$-hard for arbitrary graphs, MES is immediately $\mathcal{NP}$-hard too. Thereby, the problem was approched for different classes of graphs for which the coloring problem is in $\mathcal{P}$, in particular perfect graphs (see Golumbic (1980) for an introduction to the world of perfect graphs). Nevertheless, MES was proved $\mathcal{NP}$-hard for many of them like bipartite graphs, cographs, interval graphs and chordal graphs (Bodlaender and Jansen (1995)), permutation graphs and comparability graphs (Jansen (1998)). To the best of our knowledge, MES is polynomially solvable only with trees and forests (Baker and Coffman (1996)), split graphs (Lonc (1991)), complements of interval graphs (Bodlaender and Jansen (1995)) and proper interval graphs (Gardi (2003)). For an overview of the complexity results concerning MES, see Jansen (1998) and for other practicle applications, consult Bodlaender and Jansen (1995), Baker and Coffman (1996), Irani and Leung (1996) and Gardi (2003).

For an undirected graph G, the cardinality of a minimum coloring, or chromatic number, is denoted by $\chi(G)$. In the same way, $\chi(G, k)$ shall define the cardinality of a minimum coloring of G such that each color marks at most k vertices; note that $\chi(G, k)$, which corresponds to the length of an optimal schedule in the initial formulation, has two straightforward lower bounds: $\chi(G)$ and $\lceil n/k \rceil$. According to the previous discussions, computing $\chi(G, k)$ or optimal schedules is generally an $\mathcal{NP}$-hard problem. However, are there some interesting cases where the problem is easily solvable ? In Gardi (2003), the author establishes that *if an interval graph G admits a coloring such that each color appears at least k times, then* $\chi(G,k) = \lceil n/k \rceil$; moreover, an optimal solution is computed in linear time, given in input the graph G and its initial coloring. Such a condition is shown to be *practically* interesting because for many MES instances, simple coloring heuristics or real-life structural properties enable us to obtain it. In this note, the sufficiency of this condition is investigated for larger classes of graphs including strictly interval and proper interval graphs, namely *chordal graphs*, *complements of comparability graphs*, *circular-arc graphs*, *tolerance graphs* and *claw-free graphs*. But before giving our results, some basic definitions are recalled concerning these different classes.

2 Interval graphs and related classes

A graph $G = (V, E)$ is an *interval graph* if to each vertex $v \in V$ an interval I_v of the real line can be associated such that for each pair of distinct vertices, $uv \in E$ if and only if $I_u \cap I_v \neq \emptyset$. The family $\{I_v\}_{v \in V}$ is an *interval representation* of G. A graph G is called *proper interval graph* if there is an interval representation of G such that no interval contains properly another. The class of interval graphs coincide with the intersection of the classes of *chordal graphs* and of *complements of comparability graphs* (called shortly *co-comparability graphs*). A graph is chordal if it contains no cycle of length greater or equal than four without a chord; chordal graphs are the intersection graphs of subtrees in a tree. Comparability graphs are the transitively orientable graphs, they correspond to graphs of partial orders.

Interval graphs have two natural extensions: *circular-arc graphs* and *tolerance graphs*. Circular-arc graphs are the intersection graphs obtained from collections of arcs on a circle. Note that a *circular-arc representation* $\{A_v\}_{v \in V}$ of a graph G which fails to cover some point p on the circle is topologically the same as an interval representation of G. A graph $G = (V, E)$ is a *tolerance graph* if to each vertex $v \in V$ can be assigned an interval I_v and a positive real number t_v referred to as its tolerance, such that each pair of distinct vertices $u, v \in V$ are adjacent if and only if $|I_u \cap I_v| \geq \min\{t_u, t_v\}$. The family $\{I_v\}_{v \in V}$ is a *tolerance representation* of G. When G has a tolerance representation such that the tolerance of each vertex $v \in V$ is smaller than the length of I_v, the one is called *bounded tolerance graph*; such graphs are shown to be co-comparability graphs (Golumbic and Monma (1982)).

Finally, a graph is known as claw-free if it contains no induced copy of a tree composed of one central vertex and three leaves (commonly called a claw). One can easily observe that proper interval and circular-arc graphs cannot contain a claw, and also belong to the class of claw-free graphs.

Interval graphs, chordal graphs, comparability graphs, tolerance graphs are perfect and also are colorable in polynomial time. On the other hand, coloring circular-arc graphs or claw-free graphs is an $\mathcal{NP}$-hard problem. Besides, the recognition of all these graphs, except tolerance graphs, is done in polynomial time. For more details about these classes of graphs and their applications, the reader is referred to Golumbic (1980) and Branstadt et al. (1999). See also Figure 1 for a complete hierarchy of the classes.

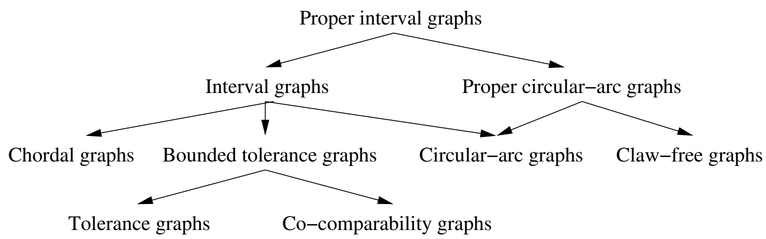

Fig. 1. The hierarchy: "class F $\rightarrow$ class G" means "class F $\subset$ class G".

3 The main results

Here some extensions of the result concerning interval graphs (Gardi (2003)) are provided for *circular-arc graphs*, *claw-free graphs* and *chordal graphs*. To show how these extensions are surprising, a very simple counterexample is given for the classes of *tolerance graphs* and of *co-comparability graphs*: $K_{k+1,k+1}$ which is the complete bipartite graph on $2k+2$ vertices partitioned into two independent sets of size $k+1$. In effect, this graph has the following *bounded tolerance representation*: define $k+1$ intervals $I_0 = [0,1], I_1 = [2,3], \ldots, I_k = [2k, 2k+1]$ with tolerances $t_0 = t_1 = \cdots = t_k = 0$ and $k+1$ intervals $I_{k+1} = I_{k+2} = \cdots = I_{2k+1} = [0, 2k+1]$ with tolerances $t_{k+1} = t_{k+2} = \cdots = t_{2k+1} = 2k+1$. It is easy to notice that $\chi(K_{k+1,k+1}, k) = 4$ since two vertices of different independent sets cannot be matched. Now, this bound is strictly larger than the lower bound $\lceil n/k \rceil = \lceil (2k+2)/k \rceil = 3$ for all $k \geq 2$.

The first positive result concerns the class of circular-arc graphs: the constructive proof given in Gardi (2003) for interval graphs is extended to obtain the following proposition.

Proposition 1. *Let G be a circular-arc graph on n vertices and k an integer. If G admits a coloring such that each color appears at least k times, then $\chi(G,k) = \lceil n/k \rceil$. Moreover, an optimal solution to* MES *is computed in linear time for a given initial coloring.*

The second proposition extends the result which holds for proper interval graphs, and circular-arc graphs according to the previous one. Moreover, an algorithmic characterization of claw-free graphs is provided through the MES problem.

Proposition 2. *An undirected graph G with n vertices is claw-free if and only if $\chi(G,k) = \max\{\chi(G), \lceil n/k \rceil\}$ for any integer k. Moreover,* MES *is solvable in $O(n^2)$ time for a given minimum coloring of G.*

The proof of this result uses chromatic exchanges (notion early developped by De Werra (1985)) coupled with suitable data structures. Finally, the next proposition starts an attempt of extension for chordal graphs.

Proposition 3. *Let G be a chordal graph on n vertices and an integer $k \leq 4$. If G admits a coloring such that each color appears at least k times, then $\chi(G,k) = \lceil n/k \rceil$.*

Three different proofs are written for the case $k = 3$. Unfortunately, these proofs are only existential and the proof for $k = 4$ becomes fastidious. However, having no counterexamples for $k \geq 5$, we conjecture that the proposition may remain valid for all k.

4 Conclusion

As conclusion, a general conjecture is proposed which includes the three previous propositions.

Conjecture 1. Let G an arbitrary graph on n vertices and k an integer. If there exists a partition $S_1, \ldots, S_q$ of G into q independent sets of size at least k such that for all pairs (S'_i, S'_j), $S'_i \subseteq S_i$ and $S'_j \subseteq S_j$, the number of edges of the bipartite graph induced by (S'_i, S'_j) is lower or equal than its number of vertices, then $\chi(G, k) = \lceil n/k \rceil$.

Circular-arc graphs, claw-free graphs and chordal graphs satisfy this property. This one can be also viewed as follows: the bipartite graph induced by (S'_i, S'_j) forms a tree with an additional edge.

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